# Convex bodies passing through holes (Computational Geometry and Discrete Mathematics)

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# Convex bodies passing through holes

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## 1. Introduction

For a given convex body, find a "small" wall hole through which the convex body can pass. This type of problems goes back to Zindler [14] in 1920, who considered a convex polytope which can pass through a fairly small circular holes. A related topic known as Prince Rupert's problem can be found in [2]. Here we concentrate on the case when the convex body is a regular tetrahedron or a regular $n$-simplex.

For a compact convex body $K \subset \mathbb{R}^n$, let $\mathrm{diam}(K)$ and $\mathrm{width}(K)$ denote the diameter and width of $K$, respectively. For $d > 0$ let $dK$ denote the convex body with diameter $d$ and homothetic to $K$. Let $S_n$, $Q_n$, and $B_n$ denote the $n$-dimensional regular simplex, the $n$-dimensional hypercube, and the $n$-dimensional ball, respectively. Thus, $1S_n$ has side length 1, $1Q_n$ has side length $1/\sqrt{n}$, and $1B_n$ has radius $1/2$.

Let $H \subset \mathbb{R}^{n-1}$ be a convex body, which we will call a hole. Let $\Pi$ be the hyperplane containing $H$, which divides $\mathbb{R}^n$ into $\Pi$ and two (open) half spaces $\Pi^+$ and $\Pi^-$. We want to push $1S_n$ from $\Pi^+$ to $\Pi^-$ through $H$. In this situation, we are interested in two types of "small" holes, namely,

$$\gamma(n,H) := \min\{d : 1S_n \text{ can pass through the hole of } dH \text{ in } \mathbb{R}^n\},$$

and

$$\Gamma(n,H) := \min\{d : 1S_n \subset (dH) \times \mathbb{R}\}.$$

Notice that $\gamma(n,H)$ and $\Gamma(n,H)$ do not depend on $\mathrm{diam}(H)$. For given $H$, we resize $H$ so that $1S_n$ can pass through the hole $H$. We will try to find a hole homethetic to $K$ with minimum diameter, which will give $\gamma$ or $\Gamma$. (Recall that $dH$ is homothetic to $H$ and $\mathrm{diam}(dH) = d$.) By definition, $1S_n$ can pass through a hole $H$ by translation perpendicular to the hyperplane containing the hole iff $\mathrm{diam}(H) \geq \Gamma(n,H)$. Thus we have $\gamma(n,H) \leq \Gamma(n,H)$.

We have $\mathrm{width}(1Q_n) = 1/\sqrt{n}$ and $\mathrm{width}(1B_n) = 1$. Steinhagen [12] determined the width of $S_n$ as follows.

$$\mathrm{width}(1S_n) = \begin{cases} \sqrt{\frac{2}{n+1}} & \text{if } n \text{ is odd,} \\ \sqrt{\frac{2n+2}{n(n+2)}} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

If $1S_n$ can pass through a hole $dH$ by translation, then

$$\mathrm{width}(dH) \geq \mathrm{width}(1S_n) = (\sqrt{2} - o(1))/\sqrt{n}. \tag{2}$$

Let $n \geq 3$. If $1S_n$ can pass through a hole $dH$, then $d \geq \mathrm{width}(1S_2) = \sqrt{3}/2$. This gives $\gamma(n,H) \geq \sqrt{3}/2$.

Brandenberg and Theobald [1] proved the following.

$$\Gamma(n,B_{n-1}) = \begin{cases} \sqrt{\frac{2(n-1)}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{\sqrt{2n(n+1)}} & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

## 2. In the 3-space

Itoh, Tanoue, and Zamfirescu [6] proved

$$\gamma(3,Q_2) = \Gamma(3,Q_2) = 1, \quad \gamma(3,B_2) = 2r = 0.8956..., \tag{4}$$

where $r \in (0,1)$ is a unique root of the equation $216x^6 - 9x^4 + 38x^2 - 9 = 0$. We note that $\gamma(3,B_2) < \Gamma(3,B_2) = 1$.

In [9], the following is proved.

$$\gamma(3,S_2) = \Gamma(3,S_2) = \frac{1+\sqrt{2}}{\sqrt{6}} = 0.9855...$$

Zamfirescu [13] proved that most convex bodies can be held by a circular frame. Using (4), one can show that a square frame of diagonal length $d$ can hold $1S_3$ iff $1/\sqrt{2} < d < 1$, and a circular frame of diameter $d$ can hold $1S_3$ iff $1/\sqrt{2} < d < \gamma(3,B_2)$, see [6].

On the other hand, it is shown in [9] that

$$\text{no triangular frame can hold a convex body.} \tag{5}$$

This is a special property for triangular frames, and in fact, we have the following.

**Theorem 1.** [9] *Every non-triangular frame holds some tetrahedron in* $\mathbb{R}^3$.

Debrunner and Mani-Levitska [3] proved that any section of a right cylinder by a plane contains a congruent copy of the base, see also [7]. This together with (5) implies the following: if a convex body, not necessarily smooth, can pass through a triangular hole, then the convex body can pass through the hole by translation perpendicular to the wall, see [9].

Itoh and Zamfirescu [5] found a hole $H \subset \mathbb{R}^2$ with $\operatorname{diam}(H) = \operatorname{width}(1S_2) = \sqrt{3}/2$ and $\operatorname{width}(H) = \operatorname{width}(1S_3) = \sqrt{2}/2$, such that $1S_3$ can pass through $H$.

## 3. Higher dimensions

3.1. **The hole $S_{n-1}$.** Recall that any plane section of a right triangular prism contains a congruent copy of a base of the prism [3, 7]. The situation in higher dimension is different. In [3], it is proved that if $n > 3$, then for any right cylinder with convex polytope base, one can find a hyperplane section which does not contain a congruent copy of the base. Nevertheless, we have the following.

**Theorem 2.** [9] *Let $K \subset \mathbb{R}^n$ be a compact convex body, and let $\Delta_{n-1}$ be a general $(n-1)$-simplex. If $K$ can pass through the hole $\Delta_{n-1}$, then this can be done by translation only.*

**Problem 1.** *Is it possible to take the translation in Theorem 2 perpendicular to the wall? Or equivalently, do $\gamma(n,S_{n-1})$ and $\Gamma(n,S_{n-1})$ coincide?*

**Theorem 3.**

$$\gamma(n,S_{n-1}) \geq \begin{cases} \sqrt{1-\frac{1}{n}} & \text{if } n \text{ is odd,} \\ \sqrt{1-\frac{1}{n+2}} & \text{if } n \text{ is even.} \end{cases}$$

*Proof.* Suppose that $1S_n$ can pass through the hole of $dS_{n-1}$. By Theorem 2, this can be done by translation only. Thus we can apply (2) with (1), which implies the desired inequality. □

The above result together with $\gamma(n,S_{n-1}) \leq \Gamma(n,S_{n-1}) \leq 1$ gives

$$\lim_{n\to\infty} \gamma(n,S_{n-1}) = \lim_{n\to\infty} \Gamma(n,S_{n-1}) = 1.$$

If the simplex does pass through a hole, then in particular the volume of some central hyperplane section of that simplex is no bigger than the volume of the hole. After the RIMS workshop, Jiří Matoušek suggested showing $\gamma(n,S_{n-1}) \to 1$ by using this simple observation. He also told us the information from Keith Ball: it is conjectured that the smallest central hyperplane section of $S_n$ is obtained by a hyperplane parallel to a facet of the simplex. According to Keith Ball's suggestion, we asked Matthieu Fradelizi about the volume of central slices of a simplex. Then, Fradelizi told us that a result in [4] implies that the volume of the smallest central hyperplane section of $S_n$ is more than $\mathrm{vol}(S_{n-1})/(2\sqrt{3})$, and this is enough for proving $\gamma(n,S_{n-1}) \to 1$.

Since the diameter of circumsphere of $1S_n$ is $\sqrt{2(n-1)/n}$, we have

$$\Gamma(n,S_{n-1})\sqrt{\tfrac{2(n-1)}{n}} \geq \Gamma(n,B_{n-1}).$$

This together with (3) implies

$$\Gamma(n,S_{n-1}) \geq \sqrt{1-\tfrac{1}{n+1}}$$

for $n$ odd. (For $n$ even, Theorem 3 gives a better lower bound for $\Gamma(n,S_{n-1})$.)

Actually $S_n$ can pass through a hole smaller than its facet.

**Theorem 4.** $\Gamma(n,S_{n-1}) < 1$ *for all* $n \geq 2$.

Let us try the case $n=3$ to get a feel. Let $S_2 = A_0A_1A_2, A_0 = (0,1/2), A_1 = (0,-1/2), A_2 = (\sqrt{3}/2,0)$, and let $\mathcal{P}$ be the right triangular prism with base

$A_0A_1A_2$. We put the unit regular tetrahedron $S_3 = B_0B_1B_2B_3$ in the prism, namely, we set

$$B_0 = (0,1/2,0), B_1 = (0,-1/2,0), B_2 = (1/\sqrt{2},0,1/2), B_3 = (1/\sqrt{2},0,-1/2).$$

Now we move the tetrahedron very slightly keeping it inside $\mathscr{P}$ so that all vertices are off the faces of $\mathscr{P}$. This can be done by rotating the tetrahedron along the $x$-axis, and push it in the direction of $x$-axis. This gives $\Gamma(3,S_2) < 1$.

3.2. **The hole $Q_{n-1}$.** In [8] the following is proved: for every $\varepsilon > 0$ there is an $N$ such that for every $n > N$ one has

$$1S_n \subset (2+\varepsilon)Q_n.$$

This gives

$$\lim_{n\to\infty} \Gamma(n,Q_{n-1}) \leq 2.$$

Clearly we have $\Gamma(n,Q_{n-1}) \geq \Gamma(n,B_{n-1})$, and we get a lower bound for $\Gamma(n,Q_{n-1})$ from (3). Here we include a simple proof of the following slightly weaker bound.

**Theorem 5.** *We have*

$$\Gamma(n,Q_{n-1}) \geq \sqrt{\frac{2(n-1)}{n+1}}, \tag{6}$$

*with equality holding iff there exists an Hadamard matrix of order $n+1$.*

*Proof.* Let $d = \Gamma(n,Q_{n-1})$. Then $1S_n$ can pass through a hole of $dQ_{n-1}$ by translation. So (2) and (1) imply

$$\mathrm{width}(dQ_{n-1}) = \frac{d}{\sqrt{n-1}} \geq \mathrm{width}(1S_n) \geq \sqrt{\frac{2}{n+1}},$$

which gives (6). Moreover, if $1S_n \subset \ell Q_n$, then we have

$$\ell \geq \frac{\sqrt{n}}{\sqrt{n-1}}\Gamma(n,Q_{n-1}) \geq \sqrt{\frac{2n}{n+1}}.$$

It is known that $\ell = \sqrt{(2n)/(n+1)}$ iff there exists an Hadamard matrix of order $n+1$, see e.g., [11]. □

**Problem 2.**

$$\gamma(n,Q_{n-1}) = \Gamma(n,Q_{n-1}) = \sqrt{2} - o(1)?$$

3.3. **The hole $B_{n-1}$.** We have $\Gamma(n, B_{n-1}) \to \sqrt{2}$ by (3). On the other hand, the following result shows $\gamma(n, B_{n-1}) \to 3/(2\sqrt{2})$. Namely, "rotation" does help for escaping from the ball hole.

**Theorem 6.** [10]

(i) *For $n$ even,*

$$\gamma(n, B_{n-1}) = \frac{3}{2\sqrt{2}}\left(1+\frac{1}{n}\right)^{-1/2} = \frac{3}{2\sqrt{2}}\left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{5}{16n^3} + O(n^{-4})\right).$$

(ii) *Let $r^2$ be a unique real root of the cubic equation*

$$8(n+1)n^3X^3 + a_2X^2 + a_1X + a_0 = 0,$$

*where*

$$\begin{aligned} a_0 &= -(9/256)(n^2-1)^2(n^4-4n^3+2n^2+4n+13), \\ a_1 &= (1/16)(n^2-1)(2n^6-6n^5-15n^4+38n^3+42n^2+48n-29), \\ a_2 &= (1/4)(8n^6-8n^5-41n^4-28n^3-10n^2+36n+27). \end{aligned}$$

*Then, for $n$ odd,*

$$\gamma(n, B_{n-1}) = 2r = \frac{3}{2\sqrt{2}}\left(1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{13}{16n^3} + O(n^{-4})\right).$$

3.4. **Hole having minimum volume.** In [5], the following problem is posed.

**Problem 3.** *Find the minimum $(n-1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that $1S_n$ can pass through it.*

The following variation seems to be easier.

**Problem 4.** *Find the minimum $(n-1)$-dimensional volume of a compact hole in a hyperplane of $\mathbb{R}^n$ such that $1S_n$ can pass through it by translation perpendicular to the hyperplane.*

We list possible candidates. Put $\sqrt{2}S_n$ in $\mathbb{R}^{n+1}$ so that the vertices are $e_1, \ldots, e_{n+1}$, where $e_i$ is the $i$-th standard base of $\mathbb{R}^{n+1}$.

Project the $\sqrt{2}S_n$ in the direction of

$$(1, -1, \overbrace{0, \ldots, 0}^{n-1}).$$

Then the hole created by the shadow has volume

$$\frac{1}{(n-1)!}\sqrt{\frac{n+1}{2}}. \tag{7}$$

Next suppose that $n$ is odd and write $n = 2k+1$. Project the $\sqrt{2}S_n$ in the direction of

$$(\overbrace{1, \ldots, 1}^{k+1}, \overbrace{-1, \ldots, -1}^{k+1}).$$

Then the corresponding hole has volume

$$\frac{2}{(n-1)!}. \tag{8}$$

Finally suppose that $n$ is even and write $n = 2k$. Project the $\sqrt{2}S_n$ in the direction of

$$(\overbrace{k+1,\ldots,k+1}^{k},\overbrace{-k,\ldots,-k}^{k+1}).$$

In this case, the volume of the hole is

$$\frac{2}{(n-1)!}\sqrt{\frac{n}{n+2}}. \tag{9}$$

Among the above examples, the smallest one is (7) for $n \leq 5$. For $n = 7$, (7) and (8) coincide. For the other cases, (8) and (9) give the smallest one.


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